

Brouwer Fixed Point Theorem

We prove Brouwer fixed point theorem using Stokes' theorem. To do this, we first need a lemma.

Let $B = B(0, 1)$ be the unit ball in \mathbb{R}^N .

Lemma. There does not exist any differentiable map $f : \bar{B} \rightarrow \bar{B}$ which leaves the boundary $\partial\bar{B}$ fixed.

Proof. Suppose, to the contrary, that f is such a map. Write $f = (f_1, \dots, f_N)$. Define the two $(N - 1)$ -forms α and β by:

$$\begin{aligned}\alpha(x) &= x_1 dx_2 \wedge \dots \wedge dx_N \\ \beta(x) &= f_1(x) df_2(x) \wedge \dots \wedge df_N(x)\end{aligned}$$

Then α and β are equal on the boundary $\partial\bar{B}$, i.e., $\iota^* \alpha = \iota^* \beta$, where ι is the canonical embedding.

Hence, $\int_{\partial\bar{B}} \iota^* \alpha = \int_{\partial\bar{B}} \iota^* \beta$.

Now, $d\alpha = dx_1 \wedge dx_2 \wedge \dots \wedge dx_N$ is the volume form.

Also, $d\beta = df_1(x) \wedge df_2(x) \wedge \dots \wedge df_N(x) = 0$ since $\dim_{\mathbb{R}} T_{f(x)}^*(\partial\bar{B}) = N - 1$, and $df_j \in T_{f(x)}^*(\partial\bar{B})$ for all j .

Therefore, applying Stokes' theorem, we get that

$\text{Volume}(\bar{B}) = \int_{\bar{B}} d\alpha = \int_{\partial\bar{B}} \iota^* \alpha = \int_{\partial\bar{B}} \iota^* \beta = \int_{\bar{B}} d\beta = 0$, an absurdity.

Hence, the lemma follows. ■

Brouwer Fixed Point Theorem. If $f : \bar{B} \rightarrow \bar{B}$ is smooth, then f has a fixed point.

Proof. Suppose, to the contrary, that f has no fixed points. Then consider the function $g : \bar{B} \rightarrow \bar{B}$ defined as follows: let $x \in \bar{B}$. Then $f(x) \neq x$ and so we can form the line segment joining $f(x)$ to x and extend it until we hit the boundary $\partial\bar{B}$. Let $g(x)$ be that point on the boundary. Note that this is an unambiguous definition. To see it analytically, we form the vector $x - f(x)$ and we need to find a nonnegative number $t \geq 0$ such that $x + t(x - f(x))$ is on the boundary. To make calculations easier, we consider the unit vector v in the direction of $x - f(x)$, i.e., $v = \frac{x - f(x)}{\|x - f(x)\|}$, and our aim is to find a $t \geq 0$ such that $g(x) = x + tv \in \partial\bar{B}$.

That is, $\|x + tv\| = 1$. Now, we solve for t .

We have $1 = \|x + tv\|^2 = \langle x + tv, x + tv \rangle = \|v\|^2 t^2 + 2 \langle x, v \rangle t + \|x\|^2 = t^2 + 2 \langle x, v \rangle t + \|x\|^2$.

Hence, we get the equation $t^2 + 2 \langle x, v \rangle t + \|x\|^2 - 1 = 0$ which has the nonnegative root

$$t = - \langle x, v \rangle + \sqrt{\langle x, v \rangle^2 + 1 - \|x\|^2}.$$

Note that $g(x) = x + tv : \bar{B} \rightarrow \bar{B}$ is a differentiable map and if $x \in \partial\bar{B}$, then $t = 0$ and so $g(x) = x$.

Thus, g is a differentiable map from \bar{B} to \bar{B} which leaves the boundary $\partial\bar{B}$ fixed.

This contradicts the above lemma and completes the proof. ■
