TEMPLE UNIVERSITY
MATHEMATICS DEPARTMENT
Complex Analysis II (Homework 4 solutions)
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1. Let $f(z)$ be analytic in $\overline{B(0;R)}$ with $f(0)=0,\ f'(0)\neq 0$, and $f(z)\neq 0$ for $0<|z|\leq R$.
Put $\rho=\min\{|f(z)|:|z|=R\}>0$. Define $g: B(0;\rho)\longrightarrow\mathbb{C}$ by:

$$g(w)=\frac{1}{2\pi i}\int\limits_{|z|=R}\frac{zf'(z)}{f(z)-w}dz$$

Show that g is analytic in $B(0;\rho)$ and that for any $w\in B(0;\rho)$, $f(g(w))=w$.

Proof. $g(w)=\frac{1}{2\pi i}\int\limits_{|z|=R}\frac{zf'(z)}{f(z)-w}dz=\frac{1}{2\pi i}\int\limits_{0}^{2\pi}\frac{Re^{it}f'(Re^{it})}{f(Re^{it})-w}iRe^{it}dt=\frac{R^2}{2\pi}\int\limits_{0}^{2\pi}\frac{e^{2it}f'(Re^{it})}{f(Re^{it})-w}dt.$
The integrand is a continuous function of t for fixed $w\in B(0;\rho)$ and an analytic function of w for fixed $t\in[0,2\pi]$. (The denominator is never 0 as we show below).
Thus, $g(w)$ is analytic in $B(0;\rho)$.
Let $w\in B(0;\rho)$ and let $h(z)=f(z)-w$. Then, both f and h have no zeroes on $|z|=R$.
(f does not have zeroes on $|z|=R$ by our assumption, and $h(z)$ does not have any zeroes on $|z|=R$ because if $h(z_0)=0$, then $f(z_0)=w$ and this is impossible since $|w|<\rho=\min\{|f(z)|:|z|=R\}$).
Also, on $|z|=R$, $|f(z)-h(z)|=|w|<\rho\leq|f(z)|$.
Thus, by Rouche's theorem, f and h have the same number of zeroes, which is 1, in $B(0;R)$.
Thus, the equation $f(z)=w$ has a unique solution in $B(0;R)$. Call this solution z_0. (i.e., $f(z_0)=w$).
Let $\phi(z)=z$. Since $h'(z)=f'(z)$, we have $g(w)=\frac{1}{2\pi i}\int\limits_{|z|=R}\frac{zf'(z)}{f(z)-w}dz=\frac{1}{2\pi i}\int\limits_{|z|=R}\frac{\phi(z)h'(z)}{h(z)}dz=\phi(z_0)=z_0.$
(Since $h(z_0)=0$ and h does not have any other zeroes or poles).
Thus, $f(g(w))=f(z_0)=w$. Hence, for any $w\in B(0;\rho)$, $f(g(w))=w$. ■

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2. Let f be a nonconstant meromorphic function on the region G. Show that neither the poles nor the zeroes of f have a limit point in G.

Proof. Let $\{p_n\}$ be a sequence of poles for f and suppose that $p_n\longrightarrow p_0\in G$.
Since G is open, we can find an $\epsilon>0: B(p_0;\epsilon)\subset G$.
But we saw in the previous homework that $f(B(p_0;\epsilon))$ is dense in $\mathbb{C}$. That is, f has a singularity at p_0.
Since this singularity is not isolated, it cannot be a pole or a removable singularity.
Thus, f cannot be meromorphic.
Let $\{z_n\}$ be a sequence of zeroes for f and suppose that $z_n\longrightarrow z_0\in G$.
z_0 cannot be a pole for f because if it was, then $\lim_{z\longrightarrow z_0}f(z)=\infty$.
But if we approach z_0 using the sequence $\{z_n\}$, we get $\lim_{z_n\longrightarrow z_0}f(z)=0$.
Thus, z_0 is not a pole. Also, since we are assuming that f is meromorphic in G, z_0 cannot be an essential singularity. Thus, f must be analytic at z_0.
Thus, we can find an $\epsilon>0$ such that $\overline{B(z_0;\epsilon)}\subset G$ and f is analytic in $\overline{B(z_0;\epsilon)}$.
Thus, f is continuous at z_0 and since $z_n\longrightarrow z_0$, $f(z_0)$ must be 0.
Being compact, we showed in a previous homework that the number of zeroes of f inside $\overline{B(z_0;\epsilon)}$ must

be finite unless $f \equiv 0$ on G.
But since $z_n \longrightarrow z_0$, we can find $N \in \mathbb{N}$ such that $z_n \in \overline{B(z_0;\epsilon)}$ for all $n \geq N$.
Thus, we have an infinite number of zeroes inside $\overline{B(z_0;\epsilon)}$.
Therefore, $f \equiv 0$ on G. A contradiction. Thus, the zeroes of f in G cannot have a limit point in G. ■

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3. Suppose f is analytic in $\overline{B(0;1)}$ and satisfies $|f(z)| < 1$ for $|z| = 1$.
(a) Find the number of solutions (counting multiplisities) of the equation $f(z) = z^n$ for $n \in \mathbb{N}$.
(b) Suppose that f is analytic in a neighborhood of $\overline{B(0;1)}$ and satisfies $|f(z)| < 1$ for $|z| = 1$.
Show that $\exists$ a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \leq 1$ for $|z| = 1$, what can you say?

Solution. **(a)** Let $g(z) = f(z) - z^n$ and $h(z) = -z^n$.
On $|z| = 1$, $|g(z)| = |f(z) - z^n| \geq |z^n| - |f(z)| = 1 - |f(z)| > 0$, and $|h(z)| = |-z^n| = |z^n| = 1$.
Therefore, both g and h do not vanish anywhere on $|z| = 1$.
Also, on $|z| = 1$, $|g(z) - h(z)| = |f(z) - z^n + z^n| = |f(z)| < 1 = |h(z)|$.
Thus, by Rouche's theorem, g and h have the same number of zeroes (counting multiplicities) in $|z| < 1$.
Therefore, since h has exactly n zeroes in $|z| < 1$, g has exactly n zeroes in $|z| < 1$.
Thus, the equation $f(z) = z^n$ has exactly n solutions in $|z| < 1$. ■
(b) By what we did in part (a), we conclude that the equation $f(z) = z$ has exactly one solution in $|z| < 1$.
That is, $\exists$ a unique z with $|z| < 1$ and $f(z) = z$.
We can say nothing if $|f(z)| \leq 1$ for $|z| = 1$ because the function
$f(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ satisfies $|f(z)| = 1$ for $|z| = 1$, $\forall\ \theta \in \mathbb{R}$ and $\forall \alpha \in B(0;1)$.
If we let $\theta = 0$ and $\alpha = 0$, then $f(z) = z$ and so $\boxed{\text{every point in the open unit disc is a solution}}$ of $f(z) = z$.
On the other hand, if we let $\theta = 0$ and $\alpha = \frac{1}{2}$, then $f(z) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z} = \frac{2z-1}{2-z}$.
Setting $f(z) = z \Longrightarrow \frac{2z-1}{2-z} = z \Longrightarrow 2z - 1 = 2z - z^2 \Longrightarrow z^2 - 1 = 0 \Longrightarrow z^2 = 1 \Longrightarrow z = \pm 1$.
That is, the equation $f(z) = z$ $\boxed{\text{does not have any solution}}$ in the open unit disc. ■

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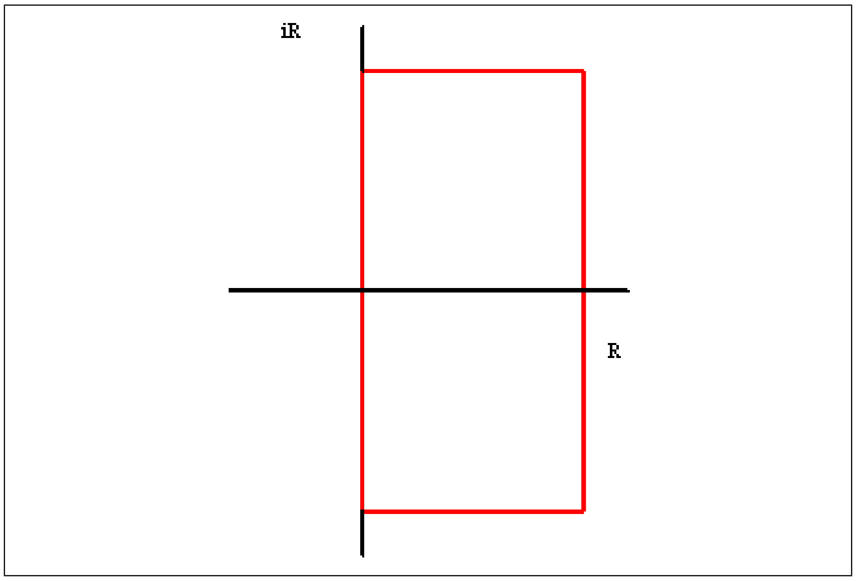
4. Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \operatorname{Re} z > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \longrightarrow 1$?

Proof. Let $R > \lambda + 1$ be given and consider the following contour C: (we don't care about orientation here)



That is, $C_R = C_1 \cup C_2 \cup C_3 \cup C_4$, where $C_1 = \{z = iy : -R \leq y \leq R\}$, $C_2 = \{z = x - iR : 0 \leq x \leq R\}$, $C_3 = \{z = R + iy : -R \leq y \leq R\}$, and $C_4 = \{z = x + iR : 0 \leq x \leq R\}$.
Now, let $f(z) = \lambda - z - e^{-z}$ and let $g(z) = \lambda - z$. Then, clearly, both f and g do not vanish on C and we have the following:

(i) On C_1, $z = iy$ for $-R \leq y \leq R$ and so $|f(z) - g(z)| = |\lambda - z - e^{-z} - \lambda + z| = |e^{-z}| = 1 < \lambda \leq |g(z)|$.
(the last inequality holds because $|g(z)| = |g(iy)| = |\lambda - iy| \geq \lambda$).
(ii) On C_2, $z = x - iR$ for $0 \leq x \leq R$ and so $|f(z) - g(z)| = |e^{-z}| = e^{-x} \leq 1 < R \leq |g(z)|$.
(the last inequality holds because $|g(z)| = |g(x - iR)| = |\lambda - x + iR| \geq |R| = R$).
(iii) On C_3, $z = R + iy$ for $-R \leq y \leq R$ and so $|f(z) - g(z)| = |e^{-z}| = e^{-R} < 1 < R - \lambda \leq |g(z)|$.
(the last inequality holds because $|g(z)| = |g(R + iy)| = |\lambda - R - iy| \geq |\lambda - R| = R - \lambda$).
(iv) On C_4, $z = x + iR$ for $0 \leq x \leq R$ and so $|f(z) - g(z)| = |e^{-z}| = e^{-x} \leq 1 < R \leq |g(z)|$.
(the last inequality holds because $|g(z)| = |g(x + iR)| = |\lambda - x - iR| \geq |-R| = R$).

Hence, $|f(z) - g(z)| < |g(z)|$ on C_R. But our choice of $R > \lambda + 1$ was arbitrary.
Thus, letting $R \longrightarrow \infty$, the strict inequality persists and by Rouche's theorem, f and g have the same number of zeroes in the half plane $\{z : \operatorname{Re} z > 0\}$.
But g has exactly one zero , $= \lambda$, in the half plane $\{z : \operatorname{Re} z > 0\}$.
Therefore, the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \operatorname{Re} z > 0\}$.

Now, let z_0 be such that $\lambda - z_0 - e^{-z_0} = 0$. Then $\overline{\lambda - z_0 - e^{-z_0}} = \overline{0} = 0$.
But $\overline{\lambda - z_0 - e^{-z_0}} = \overline{\lambda} - \overline{z_0} - \overline{e^{-z_0}} = \lambda - \overline{z_0} - e^{-\overline{z_0}}$.
Thus, $\lambda - \overline{z_0} - e^{-\overline{z_0}} = 0$ and so $\overline{z_0}$ is also a solution of the equation $\lambda - z_0 - e^{-z_0} = 0$.
Note also that if $z \in \{z : \operatorname{Re} z > 0\}$, then $\overline{z} \in \{z : \operatorname{Re} z > 0\}$.
But we proved that we have only one solution of the equation $\lambda - z - e^{-z} = 0$ in the half plane $\{z : \operatorname{Re} z > 0\}$.
Therefore, this solution must be real. ■

5. **(a)** Let $p(z)$ be a polynomial. For $R > 0$, let $M_R = \sup\{|zp(z) - 1| : |z| = R\}$. Use Rouche's theorem to show that $M_R \geq 1$.

(b) Let $0 < r < R$ and put $A = \{z : r \leq |z| \leq R\}$. Show that $\exists\ \epsilon > 0$ such that for each polynomial p,

$$\sup\{\left|p(z) - \tfrac{1}{z}\right| : z \in A\} \geq \epsilon$$

This says that $\frac{1}{z}$ is not the uniform limit of polynomials on A.

Proof. **(a)** Write out the zeroes of $p(z)$, say $z_1, z_2, ..., z_k$. Let $R > 0$ be such that $R \neq |z_n|\ \forall n = 1, 2, ..., k$.
If $M_R < 1$, then $|zp(z) - 1| < 1$ on $|z| = R$, and both $zp(z)$ and 1 do not vanish on $|z| = R$.
Thus, by Rouche's theorem, $zp(z)$ and 1 have the same number of zeroes in $B(0; R)$.
Thus, $zp(z)$ does not have any zero in $B(0; R)$. A contradiction since $zp(z)$ has 0 as a zero and $0 \in B(0; R)$.
Hence, $M_R \geq 1$. Now, if $R = |z_n|$ for some $n = 1, 2, ..., k$, then let $\delta > 0$ be such that $R - \delta > 0$ and $R - \delta \neq |z_n|\ \forall n = 1, 2, ..., k$. By what we did above, $M_{R-\delta} \geq 1$. Suppose that $M_R < 1$.
Let $g(z) = zp(z) - 1$.
Then g is entire, $M_R = \sup\{|g(z)| : |z| = R\} < 1$ and $M_{R-\delta} = \sup\{|g(z)| : |z| = R - \delta\} \geq 1$.
But by the maximum modulus principle, $\sup\{|g(z)| : |z| = R - \delta\} \leq \sup\{|g(z)| : |z| = R\}$.
Hence, $M_R \geq 1$.

(b) If $z \in A$, then $|z| \leq R$ and so $\frac{1}{|z|} \geq \frac{1}{R}$. Now,

$\sup\{\left|p(z) - \frac{1}{z}\right| : z \in A\} = \sup\{\frac{|zp(z)-1|}{|z|} : z \in A\} \geq \sup\{\frac{|zp(z)-1|}{R} : z \in A\} = \frac{1}{R}\sup\{|zp(z) - 1| : z \in A\}$
$= \frac{M_R}{R} \overset{\textbf{(a)}}{\geq} \frac{1}{R}$. Let $\epsilon = \frac{1}{R}$. ■

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6. **(a)** Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c\ \forall z \in \partial G$, then either f is a constant or f has a zero in G.
(b) Suppose that both f and g are analytic on $\overline{B(0;R)}$ with $|f(z)| = |g(z)|$ for $|z| = R$. Show that if neither f nor g vanishes in $B(0;R)$, then there is a constant λ, $|\lambda| = 1$, such that $f = \lambda g$.

Proof. **(a)** Suppose that f does not have a zero in G. Then we can apply the maximum and the minimum principles to conclude that f attains both its maximum and minimum on ∂G. But since $|f(z)| = c\ \forall z \in \partial G$, we conclude that $|f| = c$ on $\overline{G}$. Thus, by the open mapping theorem, we get that f must be constant on G. Moreover, this constant has modulus c.
(b) Let $h(z) = \frac{f(z)}{g(z)}$. Then h is analytic in $B(0;R)$ (since $g(z) \neq 0$ in $B(0;R)$), and $h(z) \neq 0$ in $B(0;R)$ (since $f(z) \neq 0$ in $B(0;R)$).
Since both f and g are analytic on $\overline{B(0;R)}$, we can find $\epsilon > 0$ such that both f and g are analytic in $B(0;R+\epsilon)$. Then h is analytic in $B(0;R+\epsilon)$ except for zeroes of g. But if we take $\overline{B(0;R+\frac{\epsilon}{2})}$, we conclude that g has at most a finite number of zeroes there.
Suppose that for some z_0 with $|z_0| = R$, $g(z_0) = 0$. Then since $|f(z)| = |g(z)|$ for $|z| = R$, we get that $f(z_0) = 0$. Also, since zeroes of analytic functions are isolated, we can find a neighborhood of z_0, $B(z_0, \delta)$, such that $g(z) \neq 0\ \forall z \in B(z_0, \delta) - \{z_0\}$.We claim that z_0 is a removable singularity of h. Note that z_0 cannot be an essential sigularity because the order of z_0 as a zero of g is finite.
If z_0 was a pole of h, then $\lim_{z \longrightarrow z_0} f(z) = \infty$.
But if we approach z_0 using a sequence of points in $B(z_0, \delta) \cap \{z : |z| = R\}$, we would get that the limit is 1 along this sequence. Thus, z_0 cannot be a pole and hence it is a removable singularity.
Hence, $h(z)$ is analytic on $\overline{B(0;R)}$, and in particular continuous on $\overline{B(0;R)}$, $|h(z)| = 1$ on $\{z : |z| = R\}$, and $h(z) \neq 0$ in $B(0;R)$. Thus, using part (a), we conclude that $h(z) = \lambda$ with $|\lambda| = 1$.
Therefore, $f = \lambda g$, for some constant λ, $|\lambda| = 1$. ■
