TEMPLE UNIVERSITY
MATHEMATICS DEPARTMENT
Complex Analysis II (Homework 2 solutions)
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1. Let $\log z = \log |\, z| + i \arg z$, $-\pi < \arg z < \pi$ be the principal branch of $\log$.
(a) Show that for any $x > 0$ and any $z \in \mathbb{C} \setminus \{x \in \mathbb{R} : x \le 0\}$, $\log xz = \log x + \log z$.
(b) Let $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Show that for any $z, w \in H$, $\log zw = \log z + \log w$.

Proof. **(a)** $\log xz \div \log |xz| + i \arg xz$. Write $z = |z|\ e^{i \arg z}$ and note that $x = |x|$ since $x > 0$.
Now, $xz = |x|\,|z|\ e^{i \arg z} = |xz|\ e^{i \arg z} \implies \boxed{\log xz = \log |xz| + i \arg z}^{(*)}$.
Now, $\log |xz| = \log\ (|x|\,|z|) = \log |x| + \log |z|$.
Thus, $(*)$ becomes $\log xz = (\log |x|) + (\log |z| + i \arg z) = (\log\ x) + (\log\ z) = \log x + \log z$.
(b) Let $z, w \in H$. Then $-\frac{\pi}{2} < \arg z,\ \arg w < \frac{\pi}{2}$ and so $-\pi < \arg z + \arg w < \pi$.
Write $z = |z|\ e^{i \arg z}$ and $w = |\, w|\ e^{i \arg w}$.
Then $zw = |z|\ e^{i \arg z} |w|\ e^{i \arg w} = |z|\ |w|\ e^{i(\arg z\ +\ \arg w)}$.
Setting $x = |z|$ and $z = |w|\ e^{i(\arg z\ +\ \arg w)}$ and applying the result in **(a)**, we get that:
$\log zw = \log\left(|z|\ |w|\ e^{i(\arg z\ +\ \arg w)}\right) \overset{\textbf{(a)}}{=} \log |z|\ + \left[\log\left(|w|\ e^{i(\arg z\ +\ \arg w)}\right)\right]$
$= \log |z|\ + [\log |w| + i(\arg z\ +\ \arg w)] = [\log |z|\ + i \arg z\ ] + [\log |w| + i \arg w] = \log z + \log w$. ■

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2. Let $f(z)$ be an entire function such that $f(z) \in \mathbb{R}\ \ \forall\ z$ with $|z| = 1$. Show that $f(z)$ is constant.

Proof. Write $f(z) = u(z) + iv(z)$ for $z \in \mathbb{C}$. Then both $u(z)$ and $v(z)$ are harmonic functions in $\mathbb{C}$.
By the given, $v(z) \equiv 0$ on the set $\partial B(0,1) = \{z \in \mathbb{C} : |z| = 1\}$.
Thus, by the maximum principle for harmonic functions,
$v(z) \equiv 0$ on the unit disc $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.
Thus, $f(z) = u(z)$ on the unit disc $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.
Hence, $f(z)$ is real-valued on the unit disc $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.
But the open mapping theorem tells us that the image of an open set under a nonconstant analytic function is an open subset of $\mathbb{C}$. But $B(0,1)$ is open and $f(B(0,1)) \subset \mathbb{R}$ and so $f(B(0,1))$ is not an open subset of $\mathbb{C}$. Thus, $f(z)$ must be a constant function. ■

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3. Let $G = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, and let $\overline{G}$ denote the closure of G in $\mathbb{C}$.
Let $f(z)$ be analytic in G and continuous on $\overline{G}$.
Suppose further that $f(z) \in \mathbb{R}$ for any z with $\operatorname{Re} z = 0$ or $\operatorname{Re} z = 1$ and for $z \in \overline{G}$, $\lim\limits_{z \longrightarrow \infty} f(z) = 0$.
Show that $f(z) = 0$.

Proof. Let $g_1(z) = iz$ and $g_2(z) = iz + 1$.
Then $g_1(z)$ is entire and it takes the real line into the imaginary axis $\{z : \operatorname{Re} z = 0\}$.
Also, $g_2(z)$ is entire and it takes the real line into the line $\{z : \operatorname{Re} z = 1\}$.
Let $\varphi_1(z) = f \circ g_1(z)$ and $\varphi_2(z) = f \circ g_2(z)$. Then :
$\cdot$ $\varphi_1(z)$ is analytic on $G_1 = \{z : -1 < \operatorname{Im} z < 0\}$, continuous on $\overline{G_1} = \{z : -1 \le \operatorname{Im} z \le 0\}$,
and $\varphi_1(z) \in \mathbb{R}$ for any $z \in \mathbb{R}$.
$\cdot$ $\varphi_2(z)$ is analytic on $G_2 = \{z : 0 < \operatorname{Im} z < 1\}$, continuous on $\overline{G_2} = \{z : 0 \le \operatorname{Im} z \le 1\}$,
and $\varphi_2(z) \in \mathbb{R}$ for any $z \in \mathbb{R}$.
Thus, by Schwartz reflection principle, we can extend both $\varphi_1(z)$ and $\varphi_2(z)$ analytically by defining:
$\varphi_1(\overline{z}) = \overline{\varphi_1(z)}$ for $z \in G_1$ and $\varphi_2(\overline{z}) = \overline{\varphi_2(z)}$ for $z \in G_2$.
Now, by the above, we have the two following facts:
$\cdot$ $f(z) = f(g_1(g_1^{-1}(z)) = \varphi_1(g_1^{-1}(z)) = \varphi_1(-iz) = \overline{\varphi_1(\overline{-iz})} = \overline{\varphi_1(iz)}$

$= \overline{f \circ g_1(iz)} = \overline{f(-z)} \quad \forall\ z \in \overline{G}.$

$\cdot\ f(z) = f(g_2(g_2^{-1}(z)) = \varphi_2(g_2^{-1}(z)) = \varphi_2(i(1-z)) = \overline{\varphi_2(\overline{i(1-z)})} = \overline{\varphi_2(i(z-1))}$
$= \overline{f \circ g_2(i(z-1))} = \overline{f(2-z)} \quad \forall\ z \in \overline{G}.$
Thus, $\overline{f(-z)} = \overline{f(2-z)} \quad \forall\ z \in G$ and so $f(z) = f(2+z) \quad \forall\ z \in \overline{G}.$
This means that f can be extended analytically to an entire function which is real on each line of the form $\{z : \operatorname{Re} z = n \in \mathbb{Z}\}$. Also, since $f(z) = f(2+z) \quad \forall\ z \in \overline{G}$, we get that $f(\mathbb{C}) = f(\overline{G})$.
Let us now show that $f(\overline{G})$ is bounded.
Since we are given that $\lim\limits_{z \longrightarrow \infty} f(z) = 0 \quad \forall\ z \in \overline{G}$, the following statement holds:
For any given $\epsilon > 0$, we can find $R > 0$ such that $|\ f(z)| < \epsilon$ for all $|z| > R$.
Let $S = \{z \in \overline{G} : -R \leq \operatorname{Im} z \leq R\}$. Then it is not hard to see that $|\ f(z)| < \epsilon$ for all $z \in \overline{G} \setminus S$.
Moreover, S is compact and so we can find $M > 0$ such that $|\ f(z)| < M$ for all $z \in S$.
Thus, $|\ f(z)| < M + \epsilon \ \ \forall\ z \in \overline{G}$.
Hence, $|\ f(z)| < M + \epsilon \ \ \forall\ z \in \mathbb{C}$ and so by Liouville's theorem, $f(z) =$ constant.
But since $\lim\limits_{z \longrightarrow \infty} f(z) = 0 \quad \forall\ z \in \overline{G}$, we conclude that this constant cannot be anything else but 0.
Therefore, $f(z) \equiv 0$. ■

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4. Let $B(0,1)$ be the open unit disc. Give an example of a nonzero analytic function in $B(0,1)$ that has infinitely many zeroes in $B(0,1)$.

First solution. Let $f(z) = \sin(\frac{z+1}{z-1})$. Then $f(z) = (\sin z) \circ \left(\frac{z+1}{z-1}\right)$ and so is analytic in $\mathbb{C} \backslash \{1\}$.
In particular, $f(z)$ is analytic in $B(0,1)$. Also, it is clear that $f(z)$ is not identically zero.
Also, $f(z) = 0 \implies \sin(\frac{z+1}{z-1}) = 0 \implies \frac{z+1}{z-1} = n\pi$, for $n \in \mathbb{Z} \implies \boxed{z = \frac{n\pi+1}{n\pi-1}, \text{ for } n \in \mathbb{Z}}$.
Now, $0 < \frac{n\pi+1}{n\pi-1} < 1 \Longrightarrow 0 < 1 + \frac{2}{n\pi-1} < 1 \Longrightarrow -1 < \frac{2}{n\pi-1} < 0 \Longrightarrow -\infty < \frac{n\pi-1}{2} < -1$
$\Longrightarrow -\infty < n\pi - 1 < -2 \Longrightarrow -\infty < n\pi < -1 \overset{n \in \mathbb{Z}}{\Longrightarrow} n \leq -1.$
Now, let $z_n = \frac{-n\pi+1}{-n\pi-1}$, $n = 1, 2, 3, \ldots.$
Then $f(z_n) = 0$ and $|z_n| = \left|\frac{-n\pi+1}{-n\pi-1}\right| < 1$ for $n = 1, 2, 3, \ldots.$
Note that the $z'_n s$ are distinct. ■

Second solution. Let $\{z_n\}$ be any sequence of points in $\mathbb{C}$ such that $z_n \overset{n \longrightarrow \infty}{\longrightarrow} \infty$.
For convenience, let us assume that $z_n \neq 0 \quad \forall\ n = 1, 2, 3, \ldots.$
Say, $z_n = |z_n|\, e^{i\theta_n}$ for $n = 1, 2, 3, \ldots.$
Since the sequence $\{\theta_n\}$ is countable and the interval $[0, 2\pi]$ is uncountable, we can find $\theta_0 \in [0, 2\pi]$ such that $\theta_0 \neq \theta_n \quad \forall\ n = 1, 2, 3, \ldots.$
Now, let $G = \mathbb{C} - \{re^{i\theta_0} : r \in [0, \infty)\}$. Then G is open and simply connected.
Thus, by the Riemann mapping theorem, we can find a bijective conformal mapping $\varphi : B(0,1) \longrightarrow G$.
Now, by Weierstrass theorem, since $z_n \overset{n \longrightarrow \infty}{\longrightarrow} \infty$, we can find an entire function $g(z)$ such that $g(z) = 0$ if and only if $z = z_n$, $n = 1, 2, 3, \ldots.$
Since g is entire, g is analytic in G and $\{z \in G : g(z) = 0\} = \{z_n\}$.
Let $f(z) = g(z) \circ \varphi(z) : B(0,1) \longrightarrow \mathbb{C}$.
Then $f(z)$ is analytic in G, $f(z)$ is not identically zero, and $f(\varphi^{-1}(z_n)) = g(z_n) = 0$.
Note that since φ is a bijection, $\varphi^{-1}(z_n) \neq \varphi^{-1}(z_m) \ \ \forall n \neq m.$ ■

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5. Let γ be a closed rectifiable curve in $B(0,1)$ and $f(z)$ a function analytic in $B(0,1)$ such that $f(z) \neq 0$ for $z \in tr(\gamma)$. Show that the set $A = \{a \in B(0,1) : f(a) = 0 \text{ and } n(\gamma, a) \neq 0\}$ is a finite set.

Proof. $tr(\gamma)$ is a compact subset of $B(0,1)$ since it is the continuous image of a compact interval.
Let $\partial B(0,1) = \{z \in \mathbb{C} : |z| = 1\}$. Then $tr(\gamma) \cap \partial B(0,1) = \varnothing$ and so by compactness of both, we get that the distance $d(tr(\gamma), \partial B(0,1)) = \delta > 0$. Thus, if we choose $\epsilon > 0$ such that $0 < \delta < \epsilon < 1$, we would get that $tr(\gamma) \subset B(0,\epsilon)$. Now, $\overline{B(0,\epsilon)}$ is compact and so the number of zeros of $f(z)$ inside $\overline{B(0,\epsilon)}$

must be finite or else, we would have an infinite number of zeroes in $\overline{B(0,\epsilon)}$ and so they must converge to a point in $\overline{B(0,\epsilon)}$ (by compactness), and this would imply by the uniqueness theorem that $f(z)$ must be identically 0. Thus, $\boxed{\text{the set } \{a \in B(0,\epsilon) : f(a) = 0 \ \} \text{ is finite}}^{(*)}$.
Now, $A = \{a \in B(0,1) : f(a) = 0 \text{ and } n(\gamma, a) \neq 0\}$
$\overset{\text{since } tr(\gamma) \subset B(0,\epsilon)}{=} \{a \in B(0,\epsilon) : f(a) = 0 \text{ and } n(\gamma,a) \neq 0\} \subset \{a \in B(0,\epsilon) : f(a) = 0 \ \} \overset{(*)}{=}$ a finite set.
Hence, A is a subset of a finite set and so it must be finite. ■

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6. Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity just state it.
(**a**) $f(z) = \frac{\sin z}{z}$ (**b**) $f(z) = \frac{\cos z}{z}$ (**c**) $f(z) = \frac{\cos z\ -\ 1}{z}$ (**d**) $f(z) = e^{\frac{1}{z}}$ (**e**) $f(z) = \frac{\log(z+1)}{z^2}$
(**f**) $f(z) = \frac{\cos(z^{-1})}{z^{-1}}$ (**g**) $f(z) = \frac{z^2+1}{z(z-1)}$ (**h**) $f(z) = \frac{1}{1-e^z}$ (**i**) $f(z) = z \sin\frac{1}{z}$ (**j**) $f(z) = z^n \sin\frac{1}{z}$.

Solution. **(a)** $\lim_{z\to 0} \frac{\sin z}{z} \overset{\text{L'Hopital}}{=} \lim_{z\to 0} \frac{\cos z}{1} = \cos 0 = 1$.
Therefore, the singularity at $z = 0$ is **removable** and we define $f(0) = 1$.

(b) $\lim_{z\to 0} \frac{\cos z}{z} = \infty$ and so there is a pole at $z = 0$.
Now, $\frac{\cos z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n)!} = \frac{1}{z} + \ \{\text{analytic function}\}$.
Therefore, the pole at $z = 0$ is a **simple pole** and the singular part is $\frac{1}{z}$.

(c) $\lim_{z\to 0} \frac{\cos z\ -\ 1}{z} \overset{\text{L'Hopital}}{=} \lim_{z\to 0} \frac{-\sin z}{1} = -\sin 0 = 0$.
Therefore, the singularity at $z = 0$ is **removable** and we define $f(0) = 0$.

(d) $f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$ and so we have an infinite number of nonzero terms in the singular part.
Therefore, the singularity at $z = 0$ is **essential**.

(e) $\lim_{z\to 0} \frac{\log(z+1)}{z^2} \overset{\text{L'Hopital}}{=} \lim_{z\to 0} \frac{1}{2z(z+1)} = \infty$ and so there is a pole at $z = 0$.
Thus, $f(z) = \frac{\log(z+1)}{z^2} = \frac{a_{-n}}{z^n} + ... + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + ...$
Now, $a_{-n} = \lim_{z\to 0} z^n f(z) = \lim_{z\to 0} \left[z^{n-2} \log(z+1)\right] = 0$ unless $n = 1$.
For $n = 1$, we get that $\lim_{z\to 0} z f(z) = \lim_{z\to 0} \left[\frac{\log(z+1)}{z}\right] \overset{\text{L'Hopital}}{=} \lim_{z\to 0} \frac{1}{(z+1)} = 1$.
That is, $f(z) = \frac{\log(z+1)}{z^2} = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + ...$
Therefore, the pole at $z = 0$ is a **simple pole** and the singular part is $\frac{1}{z}$.

(f) $f(z) = \frac{\cos(z^{-1})}{z^{-1}} = z \cos(\frac{1}{z}) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}}$ and so we have an infinite number of nonzero terms in the singular part. Therefore, the singularity at $z = 0$ is **essential**.

(g) $f(z) = \frac{z^2+1}{z(z-1)} = \frac{-2}{1-z} - \frac{1}{z} + 1 \overset{|z|<1}{=} -2\left(\sum_{n=0}^{\infty} z^n\right) - \frac{1}{z} + 1 = -\frac{1}{z} + \ \{\text{analytic function}\}$.
Therefore, the pole at $z = 0$ is a **simple pole** and the singular part is $\frac{-1}{z}$.

(h) $\lim_{z\to 0} \frac{1}{1-e^z} = \infty$ and so there is a pole at $z = 0$.
Thus, $f(z) = \frac{1}{1-e^z} = \frac{a_{-n}}{z^n} + ... + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + ...$
Now, for $n \geq 1$, $a_{-n} = \lim_{z\to 0} z^n f(z) = \lim_{z\to 0} \left[\frac{z^n}{1-e^z}\right] \overset{\text{L'Hopital}}{=} \lim_{z\to 0} \left[\frac{n z^{n-1}}{-e^z}\right] = 0$ unless $n = 1$.
For $n = 1$, we get that $\lim_{z\to 0} z f(z) = \lim_{z\to 0} \left[\frac{z}{1-e^z}\right] \overset{\text{L'Hopital}}{=} \lim_{z\to 0} \frac{1}{-e^z} = -1$.

That is, $f(z) = \frac{1}{1-e^z} = \frac{-1}{z} + a_0 + a_1 z + a_2 z^2 + ...$
Therefore, the pole at $z = 0$ is a **simple pole** and the singular part is $\frac{-1}{z}$.

(i) $f(z) = z \sin \frac{1}{z} = z \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n+1}} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n}}$ and so we have an infinite number of nonzero terms in the singular part.
Therefore, the singularity at $z = 0$ is **essential**.

(j) $f(z) = z^n \sin \frac{1}{z} = z^n \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)! z^{2m+1}} = \sum\limits_{n=0}^{\infty} \frac{(-1)^m}{(2m+1)! z^{2m-n+1}}$ and so we have an infinite number of nonzero terms in the singular part for any $n \in \mathbb{Z}$.
Therefore, the singularity at $z = 0$ is **essential** $\forall n \in \mathbb{Z}$. ■

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7. Let $f(z) = \frac{1}{z(z-1)(z-2)}$; give the Laurent expansion of $f(z)$ in each of the following annuli:
(a) ann $(0;\ 0,\ 1)$ **(b)** ann $(0;\ 1,\ 2)$ **(c)** ann $(0;\ 2,\ \infty)$.

Solution. First, we use partial fractions to get $f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$.

(a) ann $(0;\ 0,\ 1) = \{z \in \mathbb{C} : 0 < |z| < 1\}$.
Thus, $f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)} = \frac{1}{2z} + \frac{1}{1-\ z} - \frac{1}{4\left(1-\ \frac{z}{2}\right)} \overset{|z|<1}{=} \frac{1}{2z} + \left(\sum\limits_{n=0}^{\infty} z^n\right) - \frac{1}{4}\left(\sum\limits_{n=0}^{\infty} \frac{z^n}{2^n}\right)$
$= \frac{1}{2z} + \left(\sum\limits_{n=0}^{\infty} z^n\right) - \left(\sum\limits_{n=0}^{\infty} \frac{z^n}{2^{n+2}}\right) = \frac{1}{2z} + \sum\limits_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+2}}\right) z^n.$

(b) ann $(0;\ 1,\ 2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.
Thus, $f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)} = \frac{1}{2z} - \frac{1}{z(1-\frac{1}{z})} - \frac{1}{4\left(1-\ \frac{z}{2}\right)} \overset{1<|z|<2}{=} \frac{1}{2z} - \left(\frac{1}{z}\sum\limits_{n=0}^{\infty} \frac{1}{z^n}\right) - \frac{1}{4}\left(\sum\limits_{n=0}^{\infty} \frac{z^n}{2^n}\right)$
$= \frac{1}{2z} - \sum\limits_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum\limits_{n=0}^{\infty} \frac{z^n}{2^{n+2}} = ... + \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{2z} - \sum\limits_{n=0}^{\infty} \frac{z^n}{2^{n+2}}.$

(c) ann $(0;\ 2,\ \infty) = \{z \in \mathbb{C} : 2 < |z| < \infty\}$.
Thus, $f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)} = \frac{1}{2z} - \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2z\left(1-\ \frac{2}{z}\right)} \overset{|z|>2}{=} \frac{1}{2z} - \left(\frac{1}{z}\sum\limits_{n=0}^{\infty} \frac{1}{z^n}\right) + \left(\frac{1}{2z}\sum\limits_{n=0}^{\infty} \frac{2^n}{z^n}\right)$
$= \frac{1}{2z} - \left(\sum\limits_{n=0}^{\infty} \frac{1}{z^{n+1}}\right) + \left(\sum\limits_{n=0}^{\infty} \frac{2^{n-1}}{z^{n+1}}\right) = \frac{1}{2z} + \left(\sum\limits_{n=0}^{\infty} \frac{(2^{n-1}-1)}{z^{n+1}}\right) = \sum\limits_{n=2}^{\infty} \frac{(2^{n-1}-1)}{z^{n+1}}.$ ■
