Extra Question. Give a formula for $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution. Let $f(z) = \frac{\pi}{(z+a)^3 \sin(\pi z)}$, $a \notin \mathbb{Z}$. Then f is analytic in $\mathbb{C} - \{\ \{-a\} \cup \mathbb{Z}\}$. Also, f has simple poles at all the integers and f has a pole of order 3 at $z = -a$. Consider the contour given in problem#9 from HW3, where γ is the rectangular path $[n + \frac{1}{2} + ni,\ -n - \frac{1}{2} + ni,\ -n - \frac{1}{2} - ni,\ n + \frac{1}{2} - ni,\ n + \frac{1}{2} + ni]$.
We can show that $\lim_{n \longrightarrow \infty} \int_\gamma f(z)\ dz = 0$ using the exact same proof as in the homework. (Here, I will assume that you looked at my homework).

Now, by the residue theorem, $\frac{1}{2\pi i} \int_\gamma f(z)\ dz = (\mathbf{Res}\,[f, -a]) + \left(\sum_{m=-n}^{n} \mathbf{Res}\,[f, m] \right)$

$= \left(\frac{1}{2!} \lim_{z \longrightarrow -a} \frac{d^2}{dz^2} \left((z+a)^3 \frac{\pi}{(z+a)^3 \sin(\pi z)} \right) \right) + \left(\sum_{m=-n}^{n} \lim_{z \longrightarrow m} \left((z-m) \frac{\pi}{(z+a)^3 \sin(\pi z)} \right) \right)$

$= \left(\frac{\pi}{2} \lim_{z \longrightarrow -a} \frac{d^2}{dz^2} \left(\frac{1}{\sin(\pi z)} \right) \right) + \left(\sum_{m=-n}^{n} \lim_{z \longrightarrow m} \left(\frac{\pi}{(z+a)^3 \pi \cos(\pi z)} \right) \right)$

$= \left(\frac{\pi}{2} \lim_{z \longrightarrow -a} \left(\pi^2 \left[\frac{1+\cos^2(\pi z)}{\sin^3(\pi z)} \right] \right) \right) + \left(\sum_{m=-n}^{n} \left(\frac{1}{(m+a)^3 \cos(\pi m)} \right) \right)$

$= \left(\frac{\pi^3}{2} \left[\frac{1+\cos^2(-\pi a)}{\sin^3(-\pi a)} \right] \right) + \left(\sum_{m=-n}^{n} \left(\frac{1}{(m+a)^3 \cos(\pi m)} \right) \right)$

$= \left(\frac{\pi^3}{2} \left[\frac{1+\cos^2(\pi a)}{-\sin^3(\pi a)} \right] \right) + \left(\sum_{m=-n}^{n} \frac{(-1)^m}{(m+a)^3} \right)$

$= \left(\frac{-\pi^3}{2} \left[\frac{1+\cos^2(\pi a)}{\sin^3(\pi a)} \right] \right) + \left(\sum_{m=-n}^{n} \frac{(-1)^m}{(m+a)^3} \right).$

Thus, as $n \longrightarrow \infty$, we get that $0 = \frac{-\pi^3}{2} \left[\frac{1+\cos^2(\pi a)}{\sin^3(\pi a)} \right] + \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(m+a)^3}$

and so $\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(m+a)^3} = \frac{\pi^3}{2} \left[\frac{1+\cos^2(\pi a)}{\sin^3(\pi a)} \right]$.

If we let $a = \frac{1}{2}$, then we get that $\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(m+\frac{1}{2})^3} = \frac{\pi^3}{2} \left[\frac{1+\cos^2(\frac{\pi}{2})}{\sin^3(\frac{\pi}{2})} \right] = \frac{\pi^3}{2}$, and hence, $\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{16}$.
Writing out the first few negative and positive terms of this series,
$\cdots,\ m = -3 \leftrightarrow \frac{1}{5^3},\ m = -2 \leftrightarrow \frac{-1}{3^3},\ m = -1 \leftrightarrow 1,\ m = 0 \leftrightarrow 1,\ m = 1 \leftrightarrow \frac{-1}{3^3},\ m = 2 \leftrightarrow \frac{1}{5^3},\ \cdots$
we note that $\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(2m+1)^3} = 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3}$.

Thus, $\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{1}{2} \frac{\pi^3}{16} = \frac{\pi^3}{32}$.
