Duplication Formula for the Gamma Function. Show that: $\sqrt{\pi}\ \Gamma(2z) = 2^{2z-1}\ \Gamma(z)\ \Gamma(z+\frac{1}{2})$.

Proof. Let $f(z) = \frac{1}{\sqrt{\pi}}\ 2^{z-1}\ \Gamma(\frac{z}{2})\ \Gamma(\frac{z+1}{2})$. Then we have: (In the following, our domain is $(0,\infty)$).

(i) $\boxed{f(z) > 0 \text{ for } z \in (0,\infty)}^{(1)}$. This is true because it is true for Γ.

(ii) $\log f(z) = \log\left[\frac{1}{\sqrt{\pi}}\ 2^{2z-1}\ \Gamma(\frac{z}{2})\ \Gamma(\frac{z+1}{2})\right] = -\frac{1}{2}\log\pi + (2z-1)\log 2 + \log\Gamma(\frac{z}{2}) + \log\Gamma(\frac{z+1}{2})$.
Thus, $(\log f(z))'' = \left(-\frac{1}{2}\log\pi\right)'' + ((2z-1)\log 2)'' + \left(\log\Gamma(\frac{z}{2})\right)'' + \left(\log\Gamma(\frac{z+1}{2})\right)''$
$= \left(\log\Gamma(\frac{z}{2})\right)'' + \left(\log\Gamma(\frac{z+1}{2})\right)'' > 0$ for $z \in (0,\infty)$, as on page 179 of Conway.
Thus, $\boxed{\log f(z) \text{ is convex}}^{(2)}$.

(iii) $f(1) = \frac{1}{\sqrt{\pi}}\ 2^0\ \Gamma(\frac{1}{2})\ \Gamma(1) = \frac{1}{\sqrt{\pi}}.1.\sqrt{\pi}\ .1 = 1$. Thus, $\boxed{f(1) = 1}^{(3)}$.

(iv) $\forall\ z \in (0,\infty),\ zf(z) = \frac{1}{\sqrt{\pi}}\ 2^{z-1}\ z\ \Gamma(\frac{z}{2})\ \Gamma(\frac{z+1}{2}) = \frac{1}{\sqrt{\pi}}\ 2^z\ \left(\frac{z}{2}\ \Gamma(\frac{z}{2})\ \right)\Gamma(\frac{z+1}{2}) = \frac{1}{\sqrt{\pi}}\ 2^z\ \ \Gamma(\frac{z}{2}+1)\ \Gamma(\frac{z+1}{2})$.
On the other hand, $f(z+1) = \frac{1}{\sqrt{\pi}}\ 2^{(z+1)-1}\ \Gamma(\frac{z+1}{2})\ \Gamma(\frac{z+2}{2}) = \frac{1}{\sqrt{\pi}}\ 2^z\ \Gamma(\frac{z+1}{2})\ \Gamma(\frac{z}{2}+1)$.
Thus, $\boxed{zf(z) = f(z+1)}^{(4)}$.

By (1), (2), (3) and (4), Bohr-Mollerup Theorem implies that $f(z) = \Gamma(z)\ \ \forall z \in (0,\infty)$.
Thus, by the uniqueness theorem, $f(z) \equiv \Gamma(z)$.
Hence, $\Gamma(z) = \frac{1}{\sqrt{\pi}}\ 2^{z-1}\ \Gamma(\frac{z}{2})\ \Gamma(\frac{z+1}{2})$.
Substituting $2z$ for z, one gets that $\Gamma(2z) = \frac{1}{\sqrt{\pi}}\ 2^{2z-1}\ \Gamma(z)\ \Gamma(z+\frac{1}{2})$. ■
