Picard's Little Theorem. If f is nonconstant and entire, then $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C} - \{z_0\}$.

Proof. Assume that f omits a closed segment from its image, i.e., $f(\mathbb{C}) = \mathbb{C} - [-\epsilon, \epsilon]$.
Then the map $\psi(z) = \frac{1}{z+\epsilon}$ is analytic in $\mathbb{C} - \{-\epsilon\}$ and takes $\mathbb{C} - [-\epsilon, \epsilon]$ onto $G = \mathbb{C} - \{x \in \mathbb{R} : x \geq \frac{1}{2\epsilon}\}$.
Note that G is simply connected and so is conformally equivalent to the open unit disc $\mathbb{U}$.
Let $\varphi : G \longrightarrow \mathbb{U}$ be the conformal map and let $\Phi = \varphi \circ \psi \circ f : \mathbb{C} \longrightarrow \mathbb{C}$.
Then Φ is entire. Also, $|\Phi(z)| = |\varphi \circ \psi \circ f(z)| < 1$. Thus, Φ is entire and bounded and so Φ is constant.
Thus, $\varphi \circ \psi \circ f =$ constant, and since $\varphi \circ \psi$ is not constant, f must be constant.
Since $\epsilon > 0$ was arbitrary, we conclude that f "cannot miss more than one point unless it is constant". ■
Remark. Can you think of why the above proof is not complete?