Proposition 3.1.6 Let $\Omega \subset\subset \mathbb{R}^N$ have C^2 boundary and be geometrically convex. Then Ω is weakly convex.

Proof. Let ρ be a C^2 defining function for Ω, i.e., $\rho(x) < 0$ for $x \in \Omega$, $\rho(x) > 0$ for $x \in \overline{\Omega}^c$, and $\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$. To show that Ω is weakly convex, we have to show that $\forall P \in \partial\Omega$, the real Hessian, $\left(\frac{\partial^2\rho}{\partial x_j \partial x_k}(P)\right)_{1\leq j,k\leq N}$, is positive semi-definite $\forall\ w \in T_P\left(\partial\Omega\right)$.
(i.e., $\left\langle \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}(P)\right)_{1\leq j,k\leq N}, w \right\rangle = w^T \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}(P)\right)_{1\leq j,k\leq N} w = \sum_{j,k=1}^{N} \frac{\partial^2\rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0 \quad \forall\ w \in T_P\left(\partial\Omega\right)$)

So, let $P \in \partial\Omega$ and $w \in T_P\left(\partial\Omega\right)$ be given.

Define $Q : \mathbb{R} \to T_P\left(\partial\Omega\right) \subseteq \mathbb{R}^N$ by

$$Q(t) = P + tw$$

Then $Q(0) = P$, $Q'(0) = w$, and $Q''(t) = 0$ for all $t \in \mathbb{R}$.

Now, since Ω is geometrically convex, $T_P\left(\partial\Omega\right) \cap \Omega = \varnothing$. Thus, $\rho(w) \geq 0$ for all $w \in T_P\left(\partial\Omega\right)$.

Consider the function $\rho \circ Q : \mathbb{R} \to \mathbb{R}$. Then we have the following:

(i) $\left(\rho \circ Q\right)(t) \geq 0$ for all $t \in \mathbb{R}$. In particular, $\left(\rho \circ Q\right)(0) = 0$. Thus, $\rho \circ Q$ has a minimum at $t = 0$.

(ii) $\left(\rho \circ Q\right)'(0) = \nabla\rho(Q(0)) \cdot Q'(0) = \nabla\rho(P) \cdot w = 0$ since $w \in T_P\left(\partial\Omega\right)$.

(iii) $\left(\rho \circ Q\right)''(0) = \left(\left(\frac{\partial^2\rho}{\partial x_j \partial x_k}(P)\right)_{1\leq j,k\leq N} \cdot Q'(0)\right) + \underbrace{\left(\nabla\rho(P) \cdot Q''(0)\right)}_{=0 \text{ since } Q''(t)=0\ \forall t\in\mathbb{R}.}$

$$\Longrightarrow \left(\rho \circ Q\right)''(0) = \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}(P)\right)_{1\leq j,k\leq N} \cdot w = w^T \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}(P)\right)_{1\leq j,k\leq N} w = \sum_{j,k=1}^{N} \frac{\partial^2\rho}{\partial x_j \partial x_k}(P) w_j w_k.$$

Thus, to finish, we need to prove that $\left(\rho \circ Q\right)''(0) \geq 0$. This is an elementary fact from calculus:

"If $f \in C^2(\mathbb{R})$ has a minimum at $x = 0$, and $f(0) = 0$, then $f''(0) \geq 0$."

Proof. Suppose $f''(0) < 0$. Since f'' is assumed to be continuous, we can find $\epsilon > 0 : f''(x) < 0\ \forall x \in \left(-\epsilon, \epsilon\right)$.
Thus, by Taylor's theorem, in $\left(-\epsilon, \epsilon\right)$, $f(x) = f(0) + f'(0)x + \frac{f''(\xi)}{2!}x^2$ for some ξ between x and 0.
But $f(0) = 0$, and $f'(0) = 0$ (since f has a minimum at 0) $\Longrightarrow \boxed{f(x) = \frac{f''(\xi)}{2!}x^2}$ for some ξ between x and 0.
Since $f''(\xi) < 0$, we conclude that $f(x) < 0$ for all $x \in \left(-\epsilon, \epsilon\right) \setminus \{0\}$, a contradiction to the fact that f has a minimum at 0. Thus, $f''(0) \geq 0$. ■

Therefore, $\left(\rho \circ Q\right)''(0) = \sum_{j,k=1}^{N} \frac{\partial^2\rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0$, and so Ω is weakly convex. ■