Lewy's Example

Theorem. Let L denote the linear first order differential operator $L = 2\frac{\partial}{\partial \bar{z}} - 2iz\frac{\partial}{\partial t}$, where $z = x + iy \in \mathbb{C}$ and $t \in \mathbb{R}$. Let f be a continuous real valued function depending only on t. Suppose that $u = u(z,t)$ is a C^1 function satisfying $Lu = f$ in some neighborhood of the origin in $\mathbb{R}^3$. Then f is analytic at $t = 0$.

Proof. $Lu = f \Longrightarrow \boxed{\frac{\partial u}{\partial \bar{z}}(z,t) - iz\frac{\partial u}{\partial t}(z,t) = \frac{1}{2}f(t)}^{(1)}$. Suppose that u is a solution of (1) in the region $G = \{(z,t) \in \mathbb{C} \times \mathbb{R} : |z| < R \text{ and } |t| < R\}$ for some $R > 0$. Write $z = re^{i\theta}$ and set $s = r^2$.

For $|t| < R$ and $0 \le r < R$, let $V(t,r) = \int\limits_{|z|=r} u(z,t)dz = \int\limits_0^{2\pi} u(re^{i\theta},t)ire^{i\theta}d\theta = ir\int\limits_0^{2\pi} u(re^{i\theta},t)e^{i\theta}d\theta.$

Hence, $\frac{\partial V}{\partial r} = i\int\limits_0^{2\pi} u(re^{i\theta},t)e^{i\theta}d\theta + ir\int\limits_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta},t)e^{i\theta}d\theta = \mathbf{I}_1 + \mathbf{I}_2.$

· By integration by parts, $\mathbf{I}_1 = u(re^{i\theta},t)e^{i\theta}]_0^{2\pi} - \int\limits_0^{2\pi} \frac{\partial u}{\partial \theta}(re^{i\theta},t)e^{i\theta}d\theta = 0 - \int\limits_0^{2\pi} \left[\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} + \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \theta}\right] e^{i\theta}d\theta$

$= -\int\limits_0^{2\pi} \left[\frac{\partial u}{\partial z}ire^{i\theta} + \frac{\partial u}{\partial \bar{z}}(-ire^{-i\theta})\right] e^{i\theta}d\theta = -ir\int\limits_0^{2\pi} \frac{\partial u}{\partial z}e^{i2\theta}d\theta + ir\int\limits_0^{2\pi} \frac{\partial u}{\partial \bar{z}}d\theta.$

· On the other hand, $\mathbf{I}_2 = ir\int\limits_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta},t)e^{i\theta}d\theta = ir\int\limits_0^{2\pi} \left[\frac{\partial u}{\partial z}\frac{\partial z}{\partial r} + \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial r}\right] e^{i\theta}d\theta = ir\int\limits_0^{2\pi} \left[\frac{\partial u}{\partial z}e^{i\theta} + \frac{\partial u}{\partial \bar{z}}e^{-i\theta}\right] e^{i\theta}d\theta$

$= ir\int\limits_0^{2\pi} \frac{\partial u}{\partial z}e^{i2\theta}d\theta + ir\int\limits_0^{2\pi} \frac{\partial u}{\partial \bar{z}}d\theta.$

Hence, $\frac{\partial V}{\partial r} = \mathbf{I}_1 + \mathbf{I}_2 = 2ir\int\limits_0^{2\pi} \frac{\partial u}{\partial \bar{z}}d\theta = 2ir\int\limits_{|z|=r} \frac{\partial u}{\partial \bar{z}}(z,t)\frac{dz}{iz} = 2r\int\limits_{|z|=r} \frac{\partial u}{\partial \bar{z}}(z,t)\frac{dz}{z}.$

Thus, for $0 \le s < R^2$ and $|t| < R$, one has that (since $s = r^2$, $\frac{\partial V}{\partial r} = \frac{\partial V}{\partial s}\frac{\partial s}{\partial r} = 2r\frac{\partial V}{\partial s}$) :

$\frac{\partial V}{\partial s} = \frac{1}{2r}\frac{\partial V}{\partial r} = \frac{1}{2r}2r\int\limits_{|z|=r} \frac{\partial u}{\partial \bar{z}}(z,t)\frac{dz}{z} = \int\limits_{|z|=r} \frac{\partial u}{\partial \bar{z}}(z,t)\frac{dz}{z} \overset{(1)}{=} \int\limits_{|z|=r} \left[iz\frac{\partial u}{\partial t}(z,t) + \frac{1}{2}f(t)\right]\frac{dz}{z}$

$= \int\limits_{|z|=r} \left[i\frac{\partial u}{\partial t}(z,t) + \frac{1}{2z}f(t)\right]dz = \int\limits_{|z|=r} i\frac{\partial u}{\partial t}(z,t)dz + \int\limits_{|z|=r} \frac{1}{2z}f(t)dz = \left(i\int\limits_{|z|=r} \frac{\partial u}{\partial t}(z,t)dz\right) + \left(\frac{1}{2}f(t)\int\limits_{|z|=r} \frac{1}{z}dz\right)$

$= i\frac{\partial V}{\partial t}(t,r) + \frac{1}{2}f(t)\,(2\pi i) = i\left[\frac{\partial V}{\partial t}(t,r) + \pi f(t)\right].$

If one sets $F(t) = \int\limits_0^t f(\tau)d\tau$, $w = t + is$, and $U(t,s) = V(t,s) + \pi F(t)$, then one gets that $F'(t) = f(t)$ and that for $0 < s < R^2$ and $|t| < R$, $\frac{\partial U}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial U}{\partial t} + i\frac{\partial U}{\partial s}\right) = \frac{1}{2}\left(\left[\frac{\partial V}{\partial t} + \pi f(t)\right] + i\left[\frac{\partial V}{\partial s}\right]\right)$

$= \frac{1}{2}\left(\left[\frac{\partial V}{\partial t} + \pi f(t)\right] + i\left[i\left[\frac{\partial V}{\partial t} + \pi f(t)\right]\right]\right) = 0.$

Hence, U is holomorphic in the region $H = \left\{(t,s) : |t| < R, \text{ and } 0 < s < R^2\right\}$.

Moreover, U is continuous up to the line $s = 0$ since $\lim\limits_{s \to 0+} U(t,s) = V(t,0) + \pi F(t) = 0 + \pi F(t) = \pi F(t)$.

Note that $U(t,0) = \pi F(t)$ which is real valued. Hence, $U \in \mathcal{O}(H)$ and U is real on the real axis from $t = -R$ to $t = R$. Hence, by the Schwartz Reflection Principle, U extends holomorphically to the region $H^+ = \left\{(t,s) : |t| < R, \text{ and } |s| < R^2\right\}$ by defining $U(t,-s) = \overline{U(t,s)}$ for all $0 < s < R^2$.

Thus, $U(t,s)$ is holomorphic at $(0,0)$ and so $U(t,0)$ is real analytic at $t = 0$. But $U(t,0) = \pi F(t)$.

Hence, $F(t)$ is real analytic at $t = 0$ and so is $F'(t) = f(t)$. ■
